On simplified axiomatic foundations of special relativity

Sergey S. Stepanov[1]

Abstract

Year 2010 marks the 100-th anniversary since the paper by Ignatowsky was published that was devoted to the axiomatic analysis of the special theory of relativity. Despite such a long history, the ideas put forward by Ignatowsky, and later developed by Franck and Rothe, still have not earned a proper place in the physical curriculum. As a celebration of this jubilee, we would like to draw the attention of the community to logical foundations of the relativity theory. In this paper we consider the history of the field, analyse the main postulates underlying the special relativity theory, and discuss the causes that lead to fundamental physical constants arising from within the mathematical structure of the theory.

1 Introduction

It is well known that in his 1905 paper [1] Albert Einstein deduced the Lorentz transformations from the two postulates: the principle of relativity and the invariance of the speed of light. While the principle of relativity, which goes back to Galileo, had been known for a long time, the formulation of the postulate of the speed of light invariance became possible only after numerous experiments were carried out related to the Maxwell's electromagnetic theory. Einstein's simple axiomatic method had unquestionable advantages compared with those used by Henri Poincare [2] and Henrik Lorenz [3]. Einstein's article actually marked the creation of the special theory of relativity. With few exceptions [4], it is Einstein's approach that is commonly used in scientific literature when discussing the logical foundations of special relativity.

Five years after Einstein's work was published, on September 21th, 1910, Wladimir Ignatowsky delivered a talk titled "Certain general remarks on the principle of relativity" at the meeting of German naturalists and physicians. Among other things, he noted that [5, 6]:

> Now I ask myself the question: which relations or, more precisely, which equations of transformations one may obtain if one considers only the principle of relativity as the main guidance of one's research.

Ignatowsky showed that the postulate of invariance of the velocity of light was redundant and not necessary for obtaining the Lorentz transformations. In early 1911, Annalen der Physik published a work by Philipp Frank and Herman Rothe titled "On transformation of space-time coordinates from stationary to moving systems" [7], in which Ignatowsky's results were significantly developed. Within the approach of Ignatowsky, Frank and Rothe the Lorentz transformations can be deduced from the subset of axioms of classical mechanics. The fundamental constant with dimension of speed (often in the form $\alpha = 1/c^2$) arises in the theory as a result of reducing the initial information content after retracting the axiom of absolute time.

The only reference to works of Ignatowsky, Frank and Rothe in standard textbooks can be found in "Theory of Relativity" by Wolfgang Pauli [8]. Referring to the results of these studies he wrote, without detailed discussion:

> Nothing can, naturally, be said about the sign, magnitude and physical meaning of α. From the group-theoretical assumption it is only possible to derive the general form of the transformation formulae, but not their physical content.

[1] steps137@gmail.com

Pauli's book had significant influence on the subsequent physics courses, and therefore this particular evaluation of the work of Ignatowsky, Frank and Rothe did not help to make their results widely known. As can be seen from the context of the book, Pauli's skeptical attitude to their approach had most likely arisen from the fact that in group-theory derivation of Lorentz transformations, this fundamental constant does not readily get the meaning of the speed of light. However, as will see below, actually it does not have to be the speed of light.

Derivation of the Lorentz transformations without the second postulate of Einstein was rediscovered several times [4],[9]-[16]. Nevertheless, this fundamental result is practically absent from the modern phycical curriculum.

There are at least two reasons why it is important to use the approach of Ignatowsky, Frank and Rothe. First, the analysis of logical (axiomatic) foundations of any theory gives a possibility to understand it deeper and to predict the most probable ways of its further development and possible generalizations. Second, deriving Lorentz transformations without the second postulate of Einstein, as we will see, explicitly demonstrates the consistency of special relativity. Given the numerous attempts (not ceasing till the present days) to find logical errors in the conclusions of special relativity, such proof of the consistency of the foundations of the theory should play a significant role in teaching it.

Below, we will consider the system of axioms underlying the special theory of relativity. Then, we will demonstrate how the fundamental constant of speed and Lorentz transformations emerge. We shall discuss the fundamental distinction between the constant c, which is a parameter of Lorentz transformations, and the velocity of light, with which it is usually equated. Finally, certain general considerations concerning the axiomatic nature of fundamental physical theories will be expressed.

2 Axiomatics of special relativity

We will consider two inertial reference frames S and S' with parallel coordinate axes. Let the system S' move along the x axis of S system with speed v. As usual, we shall denote the time and coordinates of an event relative to S as (t, x, y, z), and the time and coordinates of the same event relative to S' as (t', x', y', z'). The Lorentz transformations establish the relation between the results of observation of a given event in two inertial reference systems (IRS). At the physical level of rigor, Lorentz transformations can be obtained from the following four postulates (axioms):

- $\mathbf{A}_1$: Transformations between IRS are described by continuous, differentiable and bijective functions.
- $\mathbf{A}_2$: If the velocities of two freely moving particles are equal in system S, they will also be equal in system S'.
- $\mathbf{A}_3$: All the inertial reference frames are equivalent.
- $\mathbf{A}_4$: The space in any IRS is isotropic.

The derivation of the transformations starts with the arbitrary functions $t' = f(t, x, y, z)$, $x' = g(t, x, y, z)$, and so on. Each axiom gradually makes the form of these functions more specific, until the initial freedom in expression disappears and the transformations take the form (we consider the one-dimensional case here):

$$x' = \frac{x - vt}{\sqrt{1 - \alpha v^2}}, \qquad t' = \frac{t - \alpha\, vx}{\sqrt{1 - \alpha v^2}}, \tag{1}$$

where α is a certain constant. In the axiomatic approach of Einstein it acquires the meaning of the inverse square of the speed of light. Using the axioms $\mathbf{A}_1$-$\mathbf{A}_4$, one can see that if $\alpha = 1/c^2 > 0$, then c has the meaning of the maximal possible invariant speed of motion of any material object.

In classical mechanics, the axiom of absolute time is added to the four axioms stated above:

- $\mathbf{A}_5$: If two events are simultaneous in system S, they are simultaneous in system S'.

With this axiom added, the value of constant α becomes equal to zero, and Lorentz transformations become Galilean transformations.

It is easy to see that axioms $\mathbf{A}_1$-$\mathbf{A}_4$ are also valid in classical mechanics. Thus, Lorentz transformations can be obtained on the basis of a subset of the axioms of Newtonian classical physics. This proves the consistency of the basic assumptions of special relativity, or, to be precise, reduces it to the proof of consistency of classical mechanics. Indeed, if some theory is consistent, then using only a subset of its axioms can not lead to contradictions.

Now we will consider the derivation of the Lorentz transformations in detail.

3 Fractional-linear transformations

A free body in any inertial reference frame moves uniformly. From this definition and the axiom $\mathbf{A}_1$ we can conclude that the most general transformations are fractional-linear functions with same denominators. For example, in one-dimensional case:

$$x' = \frac{A_1\, x + B_1\, t}{1 + D\, x + E\, t}, \qquad t' = \frac{A_2\, x + B_2\, t}{1 + D\, x + E\, t}, \tag{2}$$

where A_i, B_i, D, E are certain variables that may depend on the relative speed of two reference frames v. Moments of time $t = t' = 0$ in both systems are chosen in such a way that the origins of the frames coincide $x = x' = 0$. It is easy to verify that if a body is in uniform motion $x = x_0 + u\, t$ (that is, x_0, u are constants) relative to the system S, then its motion will also be uniform relative to S': $x' = x'_0 + u'\, t'$ (x'_0, u' are also constants).

Such fractional-linear transformations, which transform a straight line again into a straight line, are well known in projective geometry. In the present case, a straight line is the trajectory of a free particle (by definition of IRS). As far as the application to a physical situation is concerned, the possibility of fractional-linear transformations was first noted by Frank and Rothe [7]. The corresponding derivation can be found, for example, in [18, 19].

From the fractional-linear transformations one can see that equality of speeds is a relative concept. If two particles are located at different points in space and have the same speed $u = dx/dt$ in system S, then their speeds will be different in system S' (albeit constant, if motion relative to S is uniform). If one adds the axiom $\mathbf{A}_2$, the transformations become linear. They may be conveniently written as:

$$x' = \gamma(v)[x - v\, t], \qquad t' = \gamma(v)[t - \sigma(v)\, x], \tag{3}$$

where $\gamma(v)$ and $\sigma(v)$ are some unknown functions of speed, and the assumed coordination of units of time in two reference frames is such that the speed of the origin of system S' relative to S is equal to v, and the speed of the origin of S relative to S', respectively, to $-v$. In other words, the following equations hold: $x' = 0$, $x = vt$ and $x = 0$, $x' = -vt'$.

4 The principle of relativity

The third axiom, which asserts the equality of all the IRSs, plays a key role both in classical mechanics and in theory of relativity. Combined with the requirement of space isotropy $\mathbf{A}_4$, this axiom allows to establish the explicit form of transformations up to an arbitrary constant α. Let us consider three inertial reference systems S_1, S_2 and S_3, and write down the transformations between (S_1, S_2) and (S_2, S_3):

$$\begin{cases} x_2 = \gamma_1\, [x_1 - v_1\, t_1\,] \\ t_2\; = \gamma_1\, [t_1 - \sigma_1\, x_1] \end{cases} \qquad \begin{cases} x_3 = \gamma_2\, [x_2 - v_2\, t_2\,] \\ t_3\; = \gamma_2\, [t_2 - \sigma_2\, x_2], \end{cases} \tag{4}$$

where $\gamma_1 = \gamma(v_1)$, $\sigma_1 = \sigma(v_1)$, and so on, while v_1 is the speed of system S_2 relative to S_1, and v_2 – speed of system S_3 relative to S_2. Analogically, for relation between S_1 and S_3, we have:

$$\begin{cases} x_3 = \gamma_3\,[x_1 - v_3\,t_1] \\ t_3\ = \gamma_3\,[t_1 - \ \sigma_3\,x_1]. \end{cases} \tag{5}$$

Here we substitute (x_2, t_2) from the first system (4) to the second, and in place of x_3, t_3 we write the Eqs. (5):

$$\begin{cases} \gamma_3[x_1 - v_3\,t_1] = \gamma_2\gamma_1[(1 + v_2\sigma_1)\,x_1 - (v_1 + v_2)\,t_1\ ] \\ \gamma_3[t_1 - \ \sigma_3\,x_1] = \gamma_2\gamma_1[(1 + v_1\sigma_2)\,t_1\ - (\sigma_1 + \sigma_2)\,x_1]. \end{cases}$$

These equations must hold for any values of x_1 and t_1. In particular, the coefficient of x_1 term in the first equation of the system and of t_1 term in the second one must be equal:

$$\begin{cases} \gamma_3 = (1 + v_2\sigma_1)\gamma_1\gamma_2 \\ \gamma_3 = (1 + v_1\sigma_2)\gamma_1\gamma_2. \end{cases} \tag{6}$$

Excluding γ_3 we receive:

$$\frac{\sigma(v_1)}{v_1} = \frac{\sigma(v_2)}{v_2} = \alpha = const. \tag{7}$$

Since the speeds of the systems are arbitrary and independent, this equation is satisfied only if α is a constant common to all IRS and $\sigma(v) = \alpha\,v$.

Beside the composition of transformations, a reverse transformation between the two IRS must exist. The principle of equality of IRS (the principle of relativity) requires that the functional form of the reverse transformations is the same as that of straight lines, Eq. (3). Evidently, the speed of system S relative to S' is equal to $-v$, and thus, for example, coordinate transformations will look like:

$$x = \gamma(-v)[x' + v\,t']. \tag{8}$$

If we replace x', t' by the expressions from the direct transformation (3) and take into account that $\sigma(v) = \alpha\,v$, we obtain the equation for $\gamma(v)$:

$$\gamma(-v)\gamma(v) = 1/(1 - \alpha\,v^2). \tag{9}$$

It follows from the axiom of isotropy of space $\mathbf{A}_4$ that the function $\gamma(v)$ must be even: $\gamma(-v) = \gamma(v)$. For example, when measuring the length of a rod ($\Delta t = 0$), positioned along the line of relative motion of two systems, the result $\Delta x' = \gamma(v)\,\Delta x$ should not depend on the direction of motion (i.e., the sign of v).

Therefore, $\gamma(v) = 1/\sqrt{1 - \alpha\,v^2}$. We select positive sign for the square root, since for $v = 0$ we must have the unit transformation, $\gamma(0) = 1$. This results in Eqs. (1), from which the velocity-addition law follows:

$$v_3 = \frac{v_1 + v_2}{1 + \alpha\,v_1 v_2}. \tag{10}$$

This relation can be also obtained, similarly to Eq. (6), from the composition of transformations if we equate the coefficients under t_1.

5 The geometric meaning of the constant α

We have shown that the transformation between two IRSs, up to the constant α, can be derived from the axioms $\mathbf{A}_1$-$\mathbf{A}_4$. The constant α has a simple geometric meaning. Let us consider the velocity space, each point of which corresponds to the set of objects moving with the same constant velocity. The distance between two points in this space is the relative speed of objects.

Due to the principle of relativity, all points of the space of velocities should be equal. This means that the space is homogeneous and isotropic. It is a well known result in differential geometry that in this case there are only three possibilities: the flat Euclidean space, and the spaces of permanent negative and positive curvature. The first case corresponds to the Galilean rule of velocities addition and classical mechanics. The second (Lobachevsky's space) – to special relativity with $\alpha > 0$, the third (spherical geometry) occurs if $\alpha < 0$. The constant "$-\alpha$" is the curvature of the space of velocities. Such geometric interpretation once again illustrates that the relativity theory is actually contained in the principle of relativity and does not require referring to the properties of light for its foundation.

In fact, the velocity-addition law (10) is the projective transformation in Beltrami coordinates. For example, in the case of the spherical space, let us draw a tangent plane to the point of the sphere where the observer is located, and project on it the points of the hemisphere from the center of the sphere. Cartesian coordinates of the projections of points of the sphere onto the tangent plane are actually the physically measurable velocities. In the Beltrami coordinates each straight line on the sphere (the great circle arc) maps onto a Euclidean straight line on the tangent plane. If $\alpha = -1/c^2$, the radius of the sphere is equal to c, and the length s of the arc on the sphere, coming from the point of tangency is projected onto the segment of plane with length $v = c\,\tan(s/c)$. The velocities-addition law (10) is a formula for the tangent of the sum of two angles:

$$\tan(\theta_1 + \theta_2) = \frac{\tan(\theta_1) + \tan(\theta_2)}{1 - \tan(\theta_1)\tan(\theta_2)}, \tag{11}$$

where $\theta_i = s_i/c$.

In the Lobachevsky space (which corresponds to the special relativity), when the Beltrami coordinates are used, the tangent function is replaced by the hyperbolic tangent.

6 Sign of α constant

Without further assumptions it is not possible to fix the value, or even the sign, of the constant α. Various suggestions have been put forward to explain the fact that $\alpha > 0$ and to relate the constant with the speed of light. For example, Ignatowsky [5] was referring to the well-known electrodynamic phenomenon – the compression of the electric field lines of the field created by a moving charge. On the other hand, Terletskii [4] suggested to use the experimental fact of growth of mass (or, rather, relativistic energy [20]) with speed.

There are also considerations [17] that the theory with $\alpha = -1/c^2 < 0$ (spherical special relativity) is logically inconsistent. The argument is the following. From the law of velocities addition $v_3 = (v_1 + v_2)/(1 - v_1 v_2/c^2)$ in this case it follows that v_3 can be arbitrarily large, and for $v_1 v_2 = c^2$ it will become infinite. At the same time Equation (6) leads to $\gamma_3 = (1 - v_1 v_2/c^2)\gamma_1\gamma_2$ and for $v_1 v_2 > c^2$ the factor γ_3 becomes negative, which contradicts the earlier assertion that $\gamma(v) > 0$.

Nevertheless, in fact "theory of relativity" with $\alpha < 0$ has no logical contradictions. This follows in the first place from the consistency of the axioms laid into its foundation. However, from the physical point of view, it is certainly more peculiar than the conventional relativity theory. In particular, the above-noted "contradiction" occurs due to the special status of relation for velocities $v_1 v_2 = c^2$. In this case, v_3 becomes infinite, and for $v_1 v_2 > c^2$ it is formally negative. This point is no more special than $v = c$ in the conventional relativity theory, where the factor γ turns infinite. Such singularity of spherical special relativity may mean, for example, that for $v_1 < v_2$ and $v_1 v_2 > c^2$ the third system becomes invisible to the observer in the first. This has an obvious geometrical interpretation in terms of the tangent plane of Beltrami, as the points of the lower hemisphere have no projections upon it. This property of the theory looks very strange but it is nevertheless consistent, as are consistent the time dilation or relativity of simultaneity in the convenient relativity theory.

We believe that any attempt to theoretically prove that $\alpha > 0$ would not add clarity to the axiomatic foundations of the theory. In fact, there are three possibilities: $\alpha = 0$, $\alpha > 0$ and $\alpha < 0$. The first one is the limiting case of the other two, and corresponds to the classical mechanics.

From a logical point of view the third possibility is completely acceptable, however in our world it is the second one that realised, and $\alpha > 0$. This is an experimental fact, just as the very fact that α is nonzero at all.

As α is positive, it is convenient to introduce the fundamental constant with dimension of speed c, so that $\alpha = 1/c^2$. Now, when Lorentz transformations are set and their physical meaning is clear, it is easy to get all the kinematic effects of special relativity. In particular, from the law of velocity addition, it follows that the constant c has the meaning of maximal invariant speed of motion of any material object. Therefore, it is difficult to agree with the assertion made by Pauli about the absence of physical meaning of the constant α.

7 Is c the speed of light?

We believe that it is necessary to distinguish the constant c in Lorentz transformations and the speed of electromagnetic waves in vacuum. They are numerically equal but have different physical meaning.

Indeed, the special relativity is a theory that is applicable to any type of interactions, and should not require referencing the properties of a specific interaction (e.g., electromagnetic) for its justification. We do not know any law that would prohibit photon from having non-zero mass. As in the case of neutrino, possible massiveness of electromagnetic fields would not contradict the theory of relativity. Even if it turned out that there is no massless particles in our world, Lorentz transformations would not change.

We should not confuse the covariance of field theory and masslessness of the particles mediating the field interaction. Covariance implies invariance of equations (that is, the Lagrangian) with respect to the Lorentz transformations containing the fundamental constant c. Masslessness, e.g. of electrodynamics, is related to the absence of qudratic terms of the form $m^2 A^\nu A_\nu$ in the Lagrangian. Adding such terms would not violate the covariance of the theory, but would just lead to the speed of electromagnetic waves propagation being always less than the c.

Thus, the fundamental speed c has the meaning of invariant, maximally possible speed of any material bodies. The speed of light, which coincides with it numerically, is a property of a specific interaction. Its value is determined by the zero mass of a photon (less than 10^{-18} eV [21]).

In order to measure the value of the fundamental speed c, there is no need to conduct electrodynamic experiments. Any phenomenon in kinematics of the theory of relativity (time dilation, length contraction or addition of velocities [12]) allows us to measure the value of c.

8 Synchronization of time

In general, the role of light signals in foundations of special relativity is usually substantially exaggerated. For example, light signals are used for the process of time synchronization at different locations in one inertial reference system.

The procedure of synchronization is a part of the general problem of reconciling the units of time and length by different observers, located in the same or different IRSs. The specific method of measuring the length and time is not important, and different observers may use different procedures for measuring in their neighborhood. The only requirement that is imposed on them is preserving the properties of IRS.

In particular, for any measurement procedure all the free bodies must move uniformly. Then, two fixed relative to each other observers can agree on a unit of speed agreeing on the value of speed of a specific body, consistently flying at a constant speed relative to each of the observers. The motionless observers may agree upon (e.g., transfer) the units of time by periodic sending of objects with constant speed.

Note that in these conceptual experiments the value of speed of the object plays no role and is not necessarily the speed of light. Moreover, such procedures make sense if they give consistent results for *different* speeds of objects used for measurements.

The situation is similar when synchronizing the time reference point. Suppose that one observer sends a signal (an object) at time moment t_1 with constant speed u. The second observer, receiving the object at time moment T (according to his watch), sends it back with *the same* speed u. If the first observer gets the object back at a time moment t_2, the watches are supposed to be synchronized if $T = (t_1 + t_2)/2$.

Note that when the units of speed are agreed, the equality of speed u in both directions is controlled by both the observers. When synchronizing the units of speed and time, it is supposed such procedure should not depend on the specific values of speed u. In addition, the property of transitivity must be satisfied: if the observer **A** agreed the units of measurement and the time reference point with observer **B**, and **B** agreed them with **C**, then **A**'s and **C**'s measurements will also be synchronized.

So far we considered observers located within one inertial reference system. For Lorentz transformation to be meaningful, the observers from different IRS must also agree upon their units of measurement. To coordinate the units of speed, a simple agreement may be used – equality (by absolute value) of the relative speed of two systems of reference. Units of length may be synchronized in the direction perpendicular to the direction of motion. For example, this may be the distance between the trajectories of two objects moving along the direction of the relative motion. If the units of speed and length are coordinated, then units of time are also coordinated. This procedure was implied when we wrote down the Eqs. (3).

Thus, the synchronization of units does not require light signals and would remain valid when the observers use signals (objects) moving with an arbitrary constant speed.

9 The principle of parametric incompleteness

Now we return to the axioms of special relativity and once again turn our attention to the fact that Lorentz transformations are obtained from the axioms that are also valid in classical mechanics. However, unlike the classical case, the arbitrary constant – a fundamental speed c – enters the equations. What is its origin?

The axioms within a physical or mathematical theory are devised with an aim to obtain further conclusions (theorems). A system of axioms of a formal theory is called complete if any claim that can be formulated within this theory may be either proved or disproved. If we drop some of the axioms, some incompleteness inevitably arises in the conclusions of the theory. Sometimes this incompleteness may be minimal in the sense that all functional relations are still derivable, and only some parameters (constants) remain undetermined. We will refer to such case as the *parametric incompleteness*.

For example, exclusion of Euclid's fifth axiom of parallel lines leads to Lobachevsky's geometry, where a fundamental constant is present – the curvature of space. Omission of the axiom of absoluteness of time in classical mechanics leads to special relativity with fundamental constant of speed c. The principle of parametric incompleteness may thus be formulated [19]:

> If a theory describing certain subject area already exists, then in order to obtain a more general theory about the same subjects it is necessary to drop some of the axioms of the original theory. If the discarded axioms in some sense contain minimal axiomatic information, rejecting them may lead to parametrical incompleteness, and fundamental constants will appear in the theory.

The reduction of the axiom set looks as the correspondence principle, reversed. In the latter, from theory of relativity or quantum mechanics, by fixing the limit values of c and $\hbar$, we obtain classical physics. According to the principle of parametric incompleteness this procedure may be reversible, and decreasing the number of axioms one may turn classical mechanics into a more general theory. Fundamental physical constants arise as a manifestation of parametrical incompleteness of such theories.

Such deductive method for constructing new fundamental theories is very tempting and quite general. The principle of parametric incompleteness may be applied not only to analysis of the relations between Galilean and Lorentz transformations, or Euclid and Lobachevsky geometries. For example, quantum theory can be connected to rejection of the distributivity axiom in mathematical logic.

The approach based on the principle of parametric incompleteness can also produce generalizations of Lorentz transformations. For example, in the papers by Frank and Rothe [7], transformations containing three fundamental constants are derived in the class of linear functions of coordinates and time.

In the papers [22, 23] the fractional-linear (projective) generalizations of Lorentz transformations were obtained and their physical meaning analyzed. It was found [19] that these transformations have to be implemented if the three-dimensional space has constant non-zero curvature. The additional fundamental constant appearing in such generalized transformations is directly related to the curvature of space. The projective Lorentz transformations lead to a number of very interesting cosmological consequences.

10 Conclusion

The theory of relativity exists for more than 100 years. Its creation was primarily caused by the development of Maxwell's electromagnetic theory. Knowledge of the historical path leading to the theory is, of course, extremely essential. However, it rarely happens that such historical path is the most direct and that it provides rigorous logical foundation of the theory. We strongly believe that the methodological approach of deriving Lorentz transformations, going back to the work of Ignatowsky, Frank and Rothe, albeit after 100 years of neglect, must take its deserved place in the physics textbooks.

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